Inquiry-Based Learning:

An Educational Reform Based Upon Content-Centred Teaching.

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M. Padraig M. M. McLoughlin, Ph.D.
265 Lytle Hall,
Department of Mathematics,
Kutztown University of Pennsylvania
Kutztown, Pennsylvania 19530
mcloughl@kutztown.edu

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Abstract

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M. Padraig M. M. McLoughlin
Department of Mathematics,
Kutztown University of Pennsylvania

The author of this paper posits that inquiry-based learning (IBL) enacted via a modified Moore method (MMM) is a content-driven pedagogy; as such it is content–centred not instructor–centred or student–centred. The MMM is a philosophy of education where student must master material by *doing*; not simply discussing, reading, or seeing it and that authentic mathematical inquiry relies on inquiry though constructive scepticism.

We must commit to conjecture and prove or disprove said conjecture; we must do in order to learn; through real inquiry learning is achieved, and hence this paper proposes an archetype of mathematical pedagogy such that the experience of doing a mathematical argument is the reason enough for an exercise and inquiry-based learning (IBL) enacted via a modified Moore method (MMM) is an authentic way to actualise a learning environment where the content studied is the centre of the experience. The pedagogy of IBL is like no other pedagogy (for others focus on manner of exposition, recitation, activities, exercises, etc. and less with the content as oft content is secondary to the other method(s)) because in IBL content is primary. Many methods of instruction are not active but rather passive and some students wish to be passive and do the least (work) for the most (highest grade). IBL cannot be done passively. For a student to master material it is necessary for the instructor to be a master of the material so that the instructor may guide students through the content; hence, the IBL is in the tradition of a master-apprentice system. The major focus of the paper is on how the use of the MMM creates a more effective mathematical education for students; how use of the MMM established an atmosphere that created for many students firm and authentic understanding of many of the principles of mathematics; and, therefore is key in mathematics education reform.

We will also discuss how inquiry-based learning (IBL) enacted via a modified Moore method (MMM) is rooted in a philosophical position of 'positive scepticism.' What binds and supports mathematics is a search for truth, a search for what works, and a search for what is applicable within the constraints of the demand for justification. It is not the ends, but the means which matter the most - - the process at deriving an answer, the progression to the application, and the method of generalisation. These procedures demand more than mere speculative ideas; they demand reasoned and sanguine justification. Furthermore, 'positive scepticism' (or the principle of *epoikodomitikos skeptikistisis*) is meant to mean there is a demand for objectivity; an insistence on viewing a topic with a healthy dose of doubt; a requirement for remaining open to being wrong; and, a stipulation that an argument may not be constructed or built from an *a priori* perception. Hence, the nature of the process of the inquiry that justification must be supplied, analysed, and

critiqued is the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged.

So, this paper proposes a pedagogical position that deviates from the 'norm' insofar as it argues for inquiry-based learning (IBL) where the content studied is the centre of attention - - the student and the instructor should be secondary to the material in a university classroom where the experience of doing rather than witnessing is primary as it the case in an IBL-taught class.

I. Introduction, Background, and Foundation for the Idea that Inquiry-Based Learning is An Educational Reform Based Upon Content-Centred Teaching.

Mathematics has developed over the centuries through processes which include applied, computational, statistical, or theoretical inquiries. Mathematics is not unary[1] – there are many branches of mathematics, they are not mutually exclusive, nor is there but one way of creating, discovering, or doing mathematics. Mathematicians conjecture, analyse, argue, critique, prove or disprove, and can (hopefully) determine when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is a need for justification that is open to criticism and can withstand scrutiny - - there is a stated or understood demand for succinct argument from a logical foundation for the veracity of an assertion. Such a demand fits within the context of what the author refers to as positive constructive scepticism (or positive scepticism).

The author of this paper submits that we need to actively conjecture about claims; hypothesis as to their veracity or lack thereof; and, take a chance and try to prove a claim true or disprove the claim (the conjecture). We must be further willing to realise that the intuition that might have led to attempting to answer the claim in the positive or negative might be wrong; therefore, the opposite of what we thought might be true could be true. So, we allow ourselves to abandon the original attempt of trying to prove or disprove the claim and instead do the opposite of what we originally thought. We must be willing to do this process over and over until at last we have a solution (or maybe we never get a solution and are still working on the problem). Ergo, this paper is one of a sequence of papers in which the author argues the thesis that learning requires doing; only through inquiry is learning achieved, and hence the experience of doing a mathematical argument is the primary reason for the exercise (a secondary reason for an exercise is a finished product) whilst the nature of mathematical thought is one that is centred on positive constructive scepticism. 'Positive scepticism' is meant to mean demanding objectivity; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a priori* perception or position. Hence, the nature of the process of the inquiry that justification must be supplied, analysed, and critiqued is the essence of the nature of mathematical enterprise: knowledge and inquiry are inseparable and as such must be actively pursued, refined, and engaged. The author has argued [56] that not only is constructive scepticism an epistemological position as to the nature of mathematics, but it is also an axiological position for it is a value-judgement that inquiry into the nature of mathematics is a

[1] However in the U.S.A. mathematics is referred to as a unary object, "math is;" whereas elsewhere it is oft referred to as a non-unary, "maths are."

positive endeavour.[2]

It is the position of the author that inquiry-based learning (IBL) actualised by an instructor's use of a modified Moore method (MMM) is an example of what we shall call content–centred instruction (CCI) as opposed to instructor–centred instruction (ICI) or student–centred instruction (SCI). The compleat focus of the class: the students, the instructor, the materials, the book (if used), worksheets, discussion, etc. - - everything - - is centred on and driven by the <u>content</u>. Without the content there is no inquiry; without inquiry no learning; and, without learning no reason for a class at a college or university (save for a 'Paper Chase').

[2] See McLoughlin, "On the Nature of Mathematical Thought and Inquiry: A Prelusive Suggestion." Paper presented at the annual meeting of the Mathematical Association of America, Phoenix, Arizona, 2004 (ERIC Document ED 502336).

II. Content-Centred Educational Reform

It might be the case of 'splitting hairs' but it is important to define that which we mean when we use the terms student-centred instruction (SCI), instructor- centred instruction (ICI), and content-centred instruction (CCI).

We mean instruction that is student-centred when there is an active and dominant student participation component in the instruction and learning and there is little (if any) instructor participation component the instruction and learning [36, 68, 84]. Such a SCI would be group work where a class breaks into subgroups and works on projects, has discussion groups, etc. but there is little (if any) cross-discussion between or betwixt groups. SCI would be where students decide the material to be studied, set the pace of the course, work on worksheets throughout the class individually, do computer exercises throughout the class individually, etc. In each instance the instructor serves not as a guide, mentor, or tutor but as a tender, e.g.: the radical constructivist notion of knowledge created by the creator and that truth (as everything) is relative.[3,4]

We mean instruction that is instructor-centred when there is an active and dominant instructor component in the instruction and learning and there is little (if any) student participation component the instruction and learning [30]. Such an ICI would be classical German lectures, recitation, recital, narration, etc. and there is little (if any) discussion, student participation, or authentic didactic progress. ICI would be where students decide nothing, everything is laid out for them, then their 'heads are cracked opened and knowledge pour in,' e.g.: the notion of knowledge transmission, material to be studied, set the pace of the course, work on worksheets throughout the class individually, do computer exercises throughout the class individually, etc. In each instance the instructor serves not as a guide, mentor, or tutor but as a tender.[5] Such is a 'pure' example of ICI but there are nuances to the idea and such might be ideal for some courses in some disciplines.

We mean instruction that is content-centred when the material takes primacy, there is an active student participation component in the instruction

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[3]For example, see Friere, *The Politics of Education: Culture, Power, and Liberation,* 1985 or Gustein & Peterson, *Rethinking Mathematics: Teaching Social Justice by the Numbers,* 2005. Such is a 'pure' example of SCI but there are nuances to the idea and such might be acceptable for some courses within the academy; but, SCI seems to downplay the instructor's overall broader knowledge-base of the material than the students' and the instructor's overall deeper experience with the material.

[4] Exceptions could exist but if the students have a broader knowledge-base and experience with the material, then the instructor IS possibly a deterrent to learning (perhaps in a Computer Applications Course, for example or a French language conversation class where the student(s) is(are) native-speakers and the instructor is not a native speaker).

[5] I had a course where we saw the back of the instructor's head for the vast majority of the time (it was at Georgia State University and I can still remember clearly the bald spot). I had an instructor who was once interrupted by a student who attempted to ask a question, the instructor stopped lecturing, and the student was summarily thrown out of the class (it was my second day as a freshman at Emory University).

and learning and there an active instructor participation component the instruction and learning. It does not necessarily mean that class is 'controlled' $\frac{1}{2}$ of the time by students or $\frac{1}{2}$ of the time by the instructor but we do mean there is a clear give-and-take and cooperation between student and instructor but the instructor is fiducially responsible for the conduct of the class but the material is focus, the material is the base, the material is the purpose for the existence of the class and the didactic experience. Now, in CCI, the actualisation of the instruction can be somewhat student- or instructor-underscored but the content remains supreme. All interaction is through the content – the content is the focal point of the learning environment; it is the fulcrum; it is that which must be in order for the academy to be.

Content-Centred Instruction

Every question asked should be defined or inquire as to the nature of the content; every exercise should be designed to illuminate an aspect of the content or delve into material related to the content; every discussion that class undertakes should have as its purpose elucidation of an aspect of the content or material related to the content (as should any pedagogy). Therefore, it is the case that the student must *do* mathematics not be *'given'* it, just *'get'* something (from a calculator for example), or be *'witness'* to it (as is the case in some books).

Furthermore, we propose that in content-centred instruction (CCI) it is not sufficient to only intuit a solution, an argument, or an example but to construct said rigorously and as completely as possible. Therefore, CCI is not like some reformist methods where the object of a mathematics course is to "improve mathematical intuition rather than to verify it,"[6] nor is it a

[6] From the preface to a text which the author wishes not to mention for such would imply an 'attack' on the text or author which is not the objective of the quote.

pedagogy that advocates looking things up incessantly (the idea of knowing where to find something and not necessarily understanding what that something is (sort of a human Google search engine idea) where facts, knowledge, etc. are 'commodities'), nor to drum into a student facts to be regurgitated with the hope that understanding comes to the student (during the experience or after the experience).

III. Inquiry-Based Learning (IBL) Pedagogy is Content-Centred Instruction

The pedagogy of IBL is like no other pedagogy (for others focus on manner of exposition, recitation, activities, exercises, etc. and less with the content as oft content is secondary to the method) because in IBL content is primary. Many methods of instruction are not *active* but rather *passive* and some students wish to be passive and do the least (work) for the most (highest grade). IBL cannot be done passively. For a student to master material it is necessary for the instructor to be a master of the material so that the instructor may guide students through the content; hence, the IBL is in the tradition of a master-apprentice system [62]. We focus on how the use of the modified Moore method (MMM) creates a more effective mathematical inquiry-based education for students; how use of the MMM established an inquiry-based atmosphere that created for many students firm and authentic understanding of many of the principles of mathematics; and, therefore is key in mathematics education reform.

We assume that content is the central focus of any class (the traditional idea of a class where there are students and an instructor); but, is that the case? We could trace back to the Sophists versus the Socratics of ca. 400 B.C. (or most probably before) the questions of what does authentic learning mean and what is content (indeed what content is of value to study versus not)? Are there not educational schools of thought (oft traditionally harboured in a College of Education or College of Business, or a for-profit college or university, perhaps) that content is not 'all' there is to education, educational content (or a degree) is a 'commodity,' or that content is dependent upon a student or students? The literature is repleat with exemplars of arguments for primary, secondary, and higher education having purposes other than content (e.g.: civic duty, self-esteem, socialisation, preparation for employment, the 'Paper Chase,' etc.) and alludes to the existence of a strong distinctions and conflicts between "subject matter knowledge" or "content-area" and "pedagogy" or "pedagogical content knowledge."[7] In fact, there are theories of socio-mathematics, ethno-mathematics, meta-mathematics, 'mathematics for social justice,' etc.[8] Not that said is a new phenomenon, for purposes other than content can be found in primers for instruction as early as the *Ratio Studiorum*[9] of 1599 A.D. but even in that document there was much said of debate, decorum, 'patently false' ideas, etc. (but it is also clear there was concern for content). Indeed, within the field one can find the idea of content being dependent upon a student group, for example, in the Committee on the Undergraduate Program in Mathematics

[7] See, for example, Swales; Hodkinson; Rasmussen & Marrongelle; Ball, Thames, and, Phelps; or Siebert & Draper.

[8] See, for example, Stinton, et al, *Critical Mathematics Pedagogy: Transforming Teachers' Practices*, 2008 or Yackel & Cobb, "Sociomathematical Norms, Argumentation, and Autonomy in Mathematics," *Journal for Research in Mathematics Education, 27*, 1996.

[9] See, for example, Farrell, *The Jesuit Ratio Studiorum of 1599* (translation), 1970.

(CUPM) 2004 Guidelines there is much discussion of 'student audiences:' students taking general education or introductory collegiate courses in the mathematical sciences, students majoring in partner disciplines, students majoring in the mathematical sciences, and mathematical sciences majors with specific career goals, and how these audiences make instruction and content *relative* to said audience.[10] In a more problematic way there is a subtle proposition within the CUPM of 2004 Guidelines which questions the need for pre-requisites and advocates the deletion of some pre-requisites for courses for the purpose of increasing enrolment and opportunity which corresponds with the position of O'Shea & Pollatsek (1997).[11]

The author opines that to view mathematics instruction as audience dependent is an error for it relegates some or many students to a level of understanding mathematics that a Sophist would find acceptable but is not acceptable if the goal of education is a deep, full, authentic, and meaningful understanding of a topic. Furthermore, to deny content primacy is an error as is equalizing content with other considerations such as 'attitudes toward mathematics,' etc. However, to opine that instruction focuses on content but has ancillary benefits which include civic duty, self-esteem, socialisation, preparation for employment, etc. is wholly justified. For example, it has been argued[12] that IBL has added benefits of creating in students a willingness to be incorrect, to take a chance, to opine, conjecture, to be more self-reliant, and be an independent thinker. To argue that pre-requisites for a course should be reviewed and deleted if deemed unnecessary is entirely reasonable, but to advocate the deletion of pre-requisites that are antecedents to material in a course seems reckless.

There are some provocative and interesting ideas in the literature about what laity, faculty, and students should know about a subject. One particularly engaging idea is that instructors need to know more than laity and in a different way so that they can teach material and guide students. Rasmussen & Marrongelle [72] make a strong case for the idea that "pedagogical content knowledge" is an important consideration in the academy and should be a focus in teacher-education (Colleges of Education).

It appears clear from the literature that it is not agreed upon that content is the central focus of any class but for our argument we shall deem that IBL actualised by an instructor's use of a MMM in the instruction of mathematics is an example of CCI. As IBL is a content–driven pedagogy it is therefore not instructor–centred or student–centred. The MMM is a philosophy of education where student must master material by *doing*; not

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[10] See Committee on the Undergraduate Program in Mathematics, Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004, Washington, DC, Mathematical Association of America, 2004.

[11] See O'Shea and Pollatsek, "Do We Need Prerequisites?" Notices of the American Mathematical Society (1997): 44, no. 5.

[12] And is indeed being argued, for example, see Guided Discovery: Teaching Mathematics/ Transforming Lives, M. Starbird (1046-97-627) this conference (2009).

simply discussing, reading, or seeing it and that *authentic* mathematical inquiry relies on inquiry though constructive scepticism.

The idea of content-centred instruction begs the question, "what does it mean for a person to engage in authentic learning which results in understanding a topic or subject?" Many would claim to understand much because they are (in the weak case) familiar with something, have heard of it, read of it, or been told of it. Others might claim to understand much because they are (in a less weak case) been told of it by an 'authority,' read it in text on the subject, or because they habitually believe it. A person who holds either position would be a sophist. For a sophist, it is acceptable to rather than be something to seem to be something and rather than know something it suffices to seem to know a thing. The sophist values rhetoric over reason, perception over truth, and superficial knowledge of many things over deep knowledge of fewer things.[13] It seems to be the case that much of the American education system from kindergarten through pre-college is more attuned to the Sophistic rather than Socratic ideal and that much Sophistic principles and concerns are leeching into the post-secondary curriculum.

We will mean when we say that person A understands thing B (a person understands something) if and only if he is 1) able to comprehend it; to apprehend the meaning of or import of, 2) to be expert with or at by practice, 3) to apprehend clearly the character or nature of a thing, 4) to have knowledge of to know or to learn by information received, 5) to be capable of judging with knowledge, or, 6) the faculty of comprehending or reasoning.[14] Such a definition complements well Bloom's taxonomy[15] and focuses the discussion on the idea of thinking. A person can only comprehend when that person is thinking - - no thought, no understanding. So, mathematics education should be centred on encouraging a student to think for one's self: to conjecture, to analyse, to argue, to critique, to prove or disprove, and to *know* when problem is solved correctly, to *know* when an argument is valid or invalid.

Perhaps what sets mathematics apart from other disciplines in academe is the demand for succinct argument from a logical foundation for the veracity of a claim. We posit that in order for students to learn, students must be *active* in learning. Thus, the student must learn to understand a problem and solve it precisely, accurately, and correctly (not just 'get' an answer by 'any means'). The student must learn to conjecture and prove or disprove said conjecture. One cannot learn to solve problems by reading a book, we learn to solve problems by problem-solving. One cannot learn to conjecture from a book, we learn to conjecture by conjecturing! One does not learn

[13] See *The Meno* or *Gorgias* in The Dialogues of Plato Volume I, 1984, by Allen for a clear illustration of Sophism versus Socraticism.

[14] Oxford Universal Dictionary (1944), *3rd edition*, Oxford University Press: London, UK.

[15] See Sax, Principles of Educational and Psychological Measurement and Evaluation (1989) page 72.

to prove claims by reading other people's proofs in a book or on the internet or disprove claims by reading someone else's counterexample, we learn to prove or disprove claims by hard work, trying again and again until we succeed![16] There is no 'shortcut' to learning mathematics that is not inherently flawed or which carries with it the potential for non-authentic learning (sophism) nor can one teach insight; there is an element of creativity and ability to master mathematics that is not taught - - it can only be encouraged and assisted. It is perplexing that there is agreement that this basic idea of proactively practicing, engaging in hard work, and having a talent for an area is well-accepted in art, music, literature, athletics, etc. but not in mathematics education.[17] So, we submit the view that learning requires doing; only through inquiry is learning achieved; and, the experience of creating an idea and a mathematical argument to support or deny a conjecture idea is a core reason for an exercise and should be advanced above all else in mathematics education (certainly above the goal of generating a polished result, reading other people's work, or 'appreciating' mathematics without actually doing it[18]).

Indeed, when stating that students must be *active* in learning and that inquiry-based learning (IBL) enacted via a modified Moore method (MMM) is an authentic way to actualise a learning environment where the content studied is the centre of the experience, and that IBL is a content-driven pedagogy; as such it is content-centred not instructor- centred or student-centred it is meant in compleat and utter contradiction to what appears to be the 'established educational' understanding of said. To wit, "active learning is a buzz phrase that captures the teaching technique promoted by learner-centered [sic] as opposed to content-centered [sic] instruction."[19] Even the idea of *active* learning does not to be an entirely clear concept! So, it is not at all entirely clear to the author that <u>any</u> of the concepts underlying this paper: understanding, comprehension, active, learning, and authentic, etc. are well-defined and well-conceptualised ideas that are generally agreed upon. However, it is not hopeless since that upon which the topic is about, mathematics, itself is not well-defined by any means. Russell said of mathematics that it is, "the subject in which we never know what

[16]These statements are not meant to be sarcastic but to demonstrate that there is idempotency within the meaning of the words.

[17]Or education in some other areas but seemingly not in art education, music education, etc.

[18]Not that mathematics is not beautiful and the beauty of it is a wondrous thing to behold; but, to sit and watch it shouldn't be the primary goal of a mathematics education *especially at the university level.* Coaches do not have tennis students watch tennis; they have them practice tennis. Music teachers do not have piano students simply listen to recordings of Horowitz playing Chopin; they have the students practice playing the piano.

[19] See Halonen, Brown-Anderson, & McKeachie, "Teaching Thinking," in *McKeachie's Teaching Tips: Strategies, Research, and Theory for College and University Teachers* (11^{th} ed.), Boston: Houghton-Mifflin, 2002 and Yoder, & Hochevar, "Encouraging Active Learning Can Improve Students' Performance on Examinations," Teaching of Psychology 32, no. 2 (2005) page 91.

we are talking about nor whether what we are saying is true."[20]

So, the *active* learning in the inquiry-based learning (IBL) setting means that instructor presentations, expositions, recitations, etc. are minimal or non-existent for to place a student in a situation where finished, polished, or elegant solutions, arguments, proofs, etc. are presented denies the student the experience of discovery, of inquiry, of authentic academics (a 'mathematics appreciation' class). To place a student in a situation where finished solutions, arguments, proofs, etc. are unobtainable - - ever - - denies the student the pleasure in creating an end product. This means that the instructor *should* construct a carefully crafted programme so that ideas and propositions that form the basis or foundation for the students' understanding of material stems from the coursework (the content) and are initiated by the instructor and the students.

Propositions should not be from a preconceived or constructed 'project set' in a text but be such that they naturally flow from the discourse between students and the author (however, the author does direct students to problem sets in texts if it is *a propos*). These questions arise both in and out of class. The model is not the artificial 'top-down' archetype but natural 'percolation' or 'bottom-up' concept. Further, the instructor *ought* constantly monitor the progress of individual students and offer "hints," where appropriate at beginning levels. The two, experiential process and final product, cannot be disconnect. Thus, to paraphrase John Dewey, the ends and the means are the same.

Much of the undergraduate educational experience can be described as a sequence of courses that seem to some repetitious, to some disconnected, and to others of little utility. Many question the need for a core curriculum and there is not universal agreement even with a basic model for a major in mathematics. Much of the subject matter that the student studies in undergraduate mathematics <u>does connect</u> (even if the student is unaware of the connections) and build in a manner that is quite elegant. However, traditional course work oft is fraught with the problems that plague other disciplines. Much of the work the student does seem to him to be of little import. The student often does not perceive an overt connection between subjects. There are pre-requisites established between courses; but often they are arbitrary or ignored (for example in History, Political Science, or English). However, a well-crafted mathematics programme that has a logical structure to it such that pre-requisites are necessary for subsequent work, authentic content is delineated and defined, and the student can achieve true learning within the programme can be utile and meaningful ([13], [14]).

So, the author proposes that minimally IBL through the MMM is a potent method of instruction as well as a basic educational philosophy is that it creates 1) an ideal setting for later undergraduate research; 2) engages the

[20] Russell, Bertrand in an article from 1901 in *The International Monthly, 106*, page 84 as quoted in *Fundamentals of Mathematics*, Richardson (1958), New York: MacMillan, page 26.

student in synthesising material learnt from previous courses; 3) reinforces material learnt from previous courses (for the student must *use* pre-requisite material in a class taught in a modified Moore method (MMM) manner); 4) encourages the student to probe deeper into a subject; and, 5) develops the student's awareness of connections between areas of mathematics. It is opined that by engaging in said experience the student can discover whether he is truly interested in the field, and can begin to establish a sense of self so that if or when he chooses to pursue further education in mathematics he will be better prepared for post-baccalaureate study.

A superficial understanding of many subjects is an anathema to a Moore adherent; a Moore adherent craves a deep, full, and compleat (as compleat as possible) understanding of a subject (or subjects).[21] 'Coverage' of material is not a hallmark of the Moore method. On the other hand, traditional methodology includes the pace of the class set by the instructor (usually prior to the semester). 'Coverage' of material is a trademark of traditional methods. Maximal treatment of material is typical in a traditional classroom. In the IBL Calculus class as described herein neither the instructor nor student creates the syllabus – it is imposed from outside the classroom (it is standard to the college or university) – but the pace is dictated to a degree by the students and is regulated and adjusted by the instructor schedule. Hence, the author's MMM shares a commonality with traditional methods in so far as pacing is concerned; we acknowledge that not all questions can be answered and that each time a question is answered a plethora of new questions arise that may not be not answerable at the moment.

We accept the concept of minimal competency, that a student needs some skills before attempting more complex material, that is to say that there is a set of objectives that the instructor attempts to meet when teaching a class, that he is duty-bound to include that material. The goal of education is not, under the MMM methodology, 'vertical' knowledge (knowing one subject extremely well) nor 'horizontal' knowledge (knowing many subjects superfluously), but this pedagogy attempts to strike a balance between the two. Under IBL using our modified Moore method, traditional regularly administered quizzes, tests, and a comprehensive final [22] are a part of a course. However, a part of each quiz or test (no less than ten percent nor more than thirty percent) is assigned as 'take-home' so that the student may

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[21] See Davis, page 70; Fitzpatrick, "The Teaching Methods of R. L. Moore." *Higher Mathematics* 1 (1985): 44; Fitzpatrick, Some Aspects of the Work and Influence of R. L. Moore, *A Handbook of the History of Topology* 1996), page 9; Forbes, page 194; Paul R. Halmos, How To Teach. In *I Want To Be A Mathematician* (New York: Springer-Verlag, 1985), page 262; and, Edwin E. Moise, "Activity and Motivation in Mathematics." *American Mathematical Monthly* 72, 4 (1965): page 409.

[22] A comprehensive is an important part of the author's methods for it allows the student to take time to reflect on that which was learnt well, learnt, or not learnt and demonstrate a breathe of aptitude with the content rather than a depth as the presentations, quizzes, or even test might allow for demonstration.

autonomously compleat the 'take-home' portion with notes, ancillary materials, etc. whilst the rest form the 'in-class' portion of the assessment. An honour code is a part of each course the author teaches such that all graded assignments must be done by an individual and the individual confers with no one but the instructor.[23]

The MMM class includes class discussion and allows for the discussion to flow from the students but be directed by the instructor. It is should be expected that about one-fourth of many class periods are dedicated discussion of ideas about the definitions, axioms, or arguments. Approximately one-half or more of many class periods involves students presenting their work. The students present and are quizzed by the student's colleagues and by the instructor. The work presented includes solutions to problems from the book, solutions to problems from instructor created worksheets (downloaded from his web-site), solutions to problems from copies of problems from an out-of-print book (photocopied for the students), or claims made by students in which students have volunteered to solve. In the IBL class using our MMM, it is the case that the method allows for applications (minimal discussion of applications exists in the pure mathematics courses since the emphasis is on the foundations of theoretical mathematics) and modelling (with regard to the fact that students present their arguments before the class and that there exist exemplars for the students as well as the students in the class reviewing a presentation critically). The IBL class using our MMM does *not* include group assignments of any kind.

One other point about the philosophical or methodological underpinnings of inquiry-based learning (IBL) using our modified Moore method (MMM) method bears mentioning: that of personal responsibility. The students are adults and are treated as such. They are not talked down to and are treated as members of a community of scholars. Students are addressed as, "Ms. Surname" or "Mr. Surname," so that the atmosphere created is one that is professional. That the instructor has more experience is true, but that does not imply that the ideas expressed by the students have any less merit than the instructor. There are too many examples of students having ideas that were better than the author's, students who viewed a problem in a more refined manner than the author, or realised solutions to problems that the author had not worked out yet. Since the students are adults, they are held responsible to compleat work in a timely manner; but, if they do not have work completed then they are held accountable. The instructor does not do the work for the student; he leaves them to do their work.

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[23] The student signs a pledge that includes: "*N*o help from any person other than yourself and from any notes other than your own. However, you may use other books from the library. You may discuss this paper only with the instructor before handing it in to be graded. If you do not understand there directions see the grading policies under cheating. No calculators, computers, etc.

'I understand the definition of cheating and I received no help from another person nor did I confer with any other person:' <signature of student> "

In primary and secondary school the author was taught by traditionalists both in parochial and public school. The author has studied under university professors who have taught in many different ways during his formal educational experience which spans from the late 1970s to the 1990s.[24] Some of his professor include: M. F. Neff, D. Doyle, S. Batterson, and H. Sharpe of Emory University; M. Smith, S. Brown, J. Wall, B. Fitzpatrick, W. Kuperberg, P. Zenor, and C. Reed of Auburn University; and, J. Walker, Y. Hsu, C. Oshima, and J. Neel of Georgia State University. These professors taught classes or directed the research of the author using the Moore, a modified Moore, traditional, or reform methods; so, the author has been a student in each type of class. The author developed this modified Moore method (MMM) over the years of his college-level teaching experience (1982 – present) and it does has aspects of all the types of pedagogy that he was exposed to and a participant in. It is constantly being analysed, refined, and evaluated so per se it is more dynamic rather than static a system. As such it was similarly created via an action research model [39] in an empirical manner rather than in a quantitative manner.

[24] See http://faculty.kutztown.edu/mcloughl/curriculumvitae.html for a compleat curriculum vitae.

IV. Inquiry-Based Learning (IBL) versus Traditionalism or Constructivism

We argue for a pedagogical position that deviates from the 'norm' insofar as we posit for inquiry-based learning (IBL) where the content studied is the centre of attention - - the student and the instructor should be secondary to the material in a university classroom where the experience of **doing real mathematics** rather than 'busy work,' 'appreciating' mathematics, or witnessing mathematics - - **doing mathematics** is primary as it the case when in an IBL-taught class.

So, the author submits the modified Moore method (MMM) model for teaching in a manner that is inquiry-based. The basic philosophical position is 'if it works, then use it,' to paraphrase William James. The instructor must enter into the classroom without much 'baggage' - - that is to say he should be pragmatic, realistic, open to changes, revisions, and constantly assess whether or not the students are learning.

The author's MMM[25] amends several philosophical positions from the Moore, traditional, and constructivist methods. It borrows heavily from Moore's philosophy of education but relaxes several aspects of the modified Moore method. Moore's philosophy of education stated is that a person learns alone - without help or interference from others. The author's MMM philosophy of education states that a person learns best and most completely alone; *but* , sometimes needs a bit of help, encouragement, or reinforcement.

The Moore method assumes the student has a natural inquisitiveness, he must be active in learning, and as a consequent self-confidence and self-directedness is established and builds within the individual.[26]

The Moore method demands that the student not reference any texts, articles, or other materials pertaining to the course save the notes distributed by the instructor and the notes the individual takes during class. Not every student is as mature and dedicated as to be able to follow such a regulation especially in an undergraduate setting. Thus, books are not banished in a MMM classroom. The student is encouraged to use as many books as he opines is necessary to understand the material. The MMM instructor would attempt to implement this by encouraging to use books wisely; when the subject is definitions and exposition then the student should read the

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[25] Herein when referring to an instructor who uses the MMM in a classroom, sometimes the term 'MMM classroom' or "IBL classroom' will be used to refer to the classroom experience.

[26] See Davis, pages 17, 78, and 173; D. R. Forbes ,*The Texas System: R. L. Moore's Original Edition* Ph.D. dissertation (Madison, WI: University of Wisconsin, 1971), page 181; Traylor, page 13; and, Whyburn, page 354. However, the student is not always going to perform at peak efficiency given the constraints of human nature and the diversions of modern society. Therefore, our MMM assumes the natural inquisitiveness ebbs and flows or intermittently turns on or off much as a distributor cap distributes a charge in an engine; so the instructor must also act as a coach, encourager, mentor, etc. to give confidence to the student as necessary. This is not to say that such should be false; when there exists a student who (for whatever reason) really is not performing and is not progressing; then no amount of 'cheerleading' will negate the truth nor should it.

material, take notes, highlight, reread, etc. When the student is reading an exemplar argument then he should read much as Moore read in his early educational experience by keeping a page covered and exposing line after line; pausing between lines to try and devise an argument without seeing the rest of the argument, then continuing in such a manner.

Also, we should not be afraid to direct students to books or use books ourselves; but, we should train our students to use them wisely (and perhaps sparingly at times). On *rare* occasions the student is allowed to use a computer algebra system (CAS) such as *Maple* or *Mathematica* where or when such might be helpful to understand material or form a conjecture. We should not be afraid to direct students to a CAS or use a CAS ourselves; but, we should train our students to use them infrequently and wisely. We must be very careful with a CAS for it can become a crutch quickly and there are many examples of students who can push buttons, copy and paste syntax, but not understand why they are pushing the buttons, what is actually the case or not, and are very convinced that crunching 10 quadrillion examples proves, for example, a claim let us say a universal claim in $\mathbb{R}$.

This philosophy of education does not seek maximal coverage of a set amount of material, but standard competency in a given field with some depth and some breathe of understanding of material under consideration. This requires time, flexibility, patience, prudence, and precise use of language. It admits that a student cannot do everything since each great mathematician did not create *all* of mathematics. However great, mediocre, and poor mathematicians were capable of doing *some* mathematics so as such a student of mathematics must 'get his hands dirty' to use the colloquial.

Furthermore, in *Harry Potter and the Order of the Phoenix,* there is a wonderful quote where Harry encourages his classmates to press on and try again for to paraphrase, every great wizard or witch was once as the characters he addresses- - students; Harry asks his colleagues that if they could do it (learn the spells and become great witches or wizards) why couldn't they? Such is a wonderful way to view education and a great lesson. Another fortunate or fortuitous example exists in recent popular culture, the Disney film *Meet the Robinsons*, captured the sense of excitement the author attempts to create in his class and amongst his students for trying, trying again, celebrating the attempt (in the film there is a wonderful celebration <u>because</u> the character Louis fails), accepting that we are not always correct, and realising that we learn from mistakes (if we pay attention to the mistakes and analyse them), and always trying to "keep moving forward."[27] And the final popular culture reference we shall make is from the Disney film *Ratatouille*, where Remy the rat is a great chef, he hides under the character's Alfredo Linguini's hat, and he directs the kitchen to make wonderful food. Toward the end of the film it is revealed that he is the chef and all abandon

[27]A very important point to the author's MMM is to celebrate failures as well successes (in fact the failures oft lead to some great ideas and as my sainted mother, may she rest in peace, said, "we do not learn if we do not err."). It is from our failures whence we learn the most.

the kitchen; Collette (Alfredo's love interest) returns after viewing the great chef Gusteau's book, *Anyone Can Cook,* (which was the starting point for the film) and they begin cooking for a celebrated critic (Anton Ego). So, Remy directs the preparation of ratatouille and when Anton Ego tastes it he is 'sent' back to his childhood. The ratatouille is scrumptious. When Ego learns this meal was created by a rat it rocks him to his core. He learns what Chef Gusteau meant by, "anyone can cook," it is that not everyone can become a great artist but a great artist can come from anywhere.

It is this sense of excitement the author attempts to create in his class and amongst his students for trying, trying again, celebrating the attempt, accepting that we are not always correct, and realising that we learn from mistakes (if we pay attention to the mistakes and analyse them), always trying to "keep moving forward," that we (the faculty) were once as they are (students); and, a great artist (or mathematician) can come from anywhere![28] Therefore, though it may seem trite, every student in an author's MMM class receives credit in the form earning a 'board point' for attempting to solve a problem, present a solution, do a proof, etc. The students are rewarded for trying, they are rewarded for taking a chance, they are encouraged to expose their work to their peers and stand up and defend their ideas and their work. Said points add into the student's total points at the end of the semester and there are a plethora of example where said points produced the 'rewarding' result of a student's grade being positively impacted.

Much of the points that highlight the strengths of the modified Moore method may be summed up as the MMM *accents, celebrates, encourages, and attempts to hone an internal locus of control*. The traditional scheme there does not appear to be as prevalent a focus on the internal; indeed, rather there is a clear focus on ideas from the external (the instructor, a calculator, a computer, or the book). The constructivist scheme there does not appear to be as prevalent a focus on the internal; indeed, rather there seems to be a focus on ideas from the external (the group, a calculator, a computer, or the book) and the internal and individual are not primary.

The Moore method demands that the students not collaborate. Moore stated this position clearly:

> I don't want any teamwork. Suppose some student goes to the board.
> Some other student starts to make suggestions. Suppose some
> how or another a discussion begins to start. One person suggests
> something, then another suggests something else...
> after all this discussion suppose somebody finally gets a theorem...
> who's is it? He'd [the presenter] want a theorem to be his - he'd

[28]But, that not everyone can be a great mathematician.

want a theorem, not a joint product![29]

Our MMM eases the position Moore proposed and demands no collaboration on material *before* student presentations and no collaboration on any graded assignment (more on this later) and requests minimal collaboration on material after student presentations. *After* student presentations, if a student does not understand a part of an argument or nuance of said argument, the students are permitted to discuss the argument as well as devise other arguments.

The Moore method does not include subject lectures. Our MMM includes minimal lectures before student presentations over definitions and terminology, an occasional exemplar argument (especially early in an undergraduate mathematics programme (more on this later)), as well as subsequent lectures after the students discuss the work(s) presented when the instructor finds there is confusion or misunderstanding about the material amongst the students. However, the MMM method is not as 'lecture heavy' as a traditional class because under the MMM we seek to enable and advance inquiry amongst the students so, the instructor does not enter the class begin lecturing and only end recitation at the end of the period.

Under the MMM, everything *should* be defined, axiomatised, or proven based on the definitions and axioms whether in class or referenced. In this regard the MMM is reminiscent of Wilder's axiomatic methods [88] - [91]. Everything *cannot* be defined, discussed, etc. within class; hence, the allowance for some reference material. Indeed, the MMM avails itself of a trite example of the technology of the 21^{st} century; thus, additional class materials are available for students to download from an instructor created web-site. These handouts have several purposes including delving deeper into a subject; clarifying material in a text; correcting a text used in the class; clarifying a discussion that was interrupted in the class by the end of the class; correcting a claim that was made in class that was in error; reaction to work students did (usually on a quiz or a test); or, posing several additional problems or questions in the form of additional exercises for the students to ponder. Moreover, the handouts present students with material previously discussed, claims which were made during the class (by students or the instructor), and conjectures that were not presented by students in the class along with proposed arguments as to the veracity of the claims. The students critically read the proposed arguments and note whether or not the proposed solution is correct. Thus, the MMM includes *potentially* more reading of mathematics materials than the Moore method, though perhaps less than the traditional method.

We must also acknowledge that the undergraduate experience is bereft

[29] R. L. Moore, *Challenge in the Classroom* (Providence, RI: American Mathematical Society, 1966), videocassette.

with time constraints. Neither the instructor nor student creates the schedule. That is imposed from outside the classroom. The pace is dictated to a degree by the students but is regulated and adjusted by the instructor. Hence, the MMM shares a commonality with traditional methods in so far as pacing is concerned; the MMM acknowledges that not all questions can be answered and that each time a question is answered a plethora of new questions arise that may not be not answerable at the moment. Therefore, the MMM seeks to balance the question of 'how to' with the question of 'why.' A subject that is founded upon axioms and is developed from those axioms concurrently can be addressed with the questions 'why' and 'how to.' The students have a *reasonable* amount of time to work with the material, and more than that set of objectives is met each semester. Said reasonable amount of time varies from semester to semester dependent on conditions of the class such as background of the students, length of the semester, and other additional conditions that are not predictable before the class commences.

Traditional methods include regularly administered quizzes, tests, and finals. The MMM also includes said assessments. However, a part of each quiz or test (no less than ten percent nor more than thirty percent) is assigned as 'take-home' so that the student may autonomously compleat the 'take-home' portion with notes, ancillary materials, etc. whilst the rest form the 'in-class' portion of the assessment. Ideally the true nature of work is with time allowance; but also includes time constraints. Hence, an artificial time constraint of a class period is imposed upon the instructor and students because of the system in which they work. However, not all meaningful and educationally enriching exercise can be included on a test in a class period; hence, the inclusion of take-home assessments. However, not all assessments should be take home since there should be some measure of retention of key concepts by the student and if all were take home (or open book in class assessment) then the exercise is perhaps more about finding information than retaining it. On the other hand, if all assignments were in-class then one could argue that the exercises can and oft deteriorate into students regurgitating trite tid-bits and small parts of concepts rather than engaging in deeper analysis. Nonetheless, practical considerations force the MMM instructor to note that if all assignments were take-home; there would be a contingent of students who cheat - - there is no way around this sad fact of modern society and the reality that ethics are in flux; so the MMM instructor is duty-bound to attempt to make the educational experience fair (or as fair as is within his control and as fair as is humanly possible).

At least one quiz is administered through three to six class periods, part in class part take home, or all take home in which the students are asked to prove or disprove conjectures. They are required (of course) to work alone. The quizzes are graded and commentary included so that feedback is more than just a grade. Also, there are three or four major tests during the semester and a comprehensive final; thus, the MMM is grading intensive for the

instructor. The frequency of the quizzes creates a standard for the students so they do not fall behind. The final and the tests gives the students the ability to demonstrate competency or proficiency[30] over a part of the course and an opportunity and responsibility to digest and synthesise the material. The testing schedule differs from the Moore and reform method and shares a commonality with the traditional method. It may be a tad more 'quiz intensive' than traditional methods, but the author has found that many of his colleagues who employ traditional methods grade homework (which is not a part of the MMM) so it might be similar to the traditional methods in that regard.

Experience with many different course sizes over the past twenty years has led the author to conclude that optimal course size is between approximately fifteen and twenty-five. When there are less than about fifteen students, then the class discussions often suffer for a lack of interaction. When the class size is more than about twenty-five students, then class discussions are often difficult to facilitate and can be problematic because so many students wish to be heard simultaneously. Also, if the class size exceeds approximately twenty-five, then the burden of grading so many papers becomes quite heavy and the turn around time lengthens which is detrimental. It seems that it is best to provide feedback in a timely manner so that the students have time to reflect on their work and discuss the work in follow-up session during office hours. If too much time has elapsed between the times students hand the papers in and they get the papers back, their memory of *w*hy they thought what they thought dwindles and the educational experience for the student suffers.

Many policies of the department supersede that of the instructor. For example, in amount of material discussed, class hours, text, etc. the instructor may not be able to control these considerations but is a part of a faculty such that compromises become necessary and flexibility a must. For example, ideally courses would be four semester or five semester hour courses, but are not at Kutztown University of Pennsylvania (KUP). *All* mathematics courses are three semester hour courses.[31] Additionally, when teaching these courses it may be the case that a student takes one or the other with an instructor who does not use the MMM, but uses traditional or reform methods,[32] so some students may be accustomed to the method and are more tractable in the some courses than others. As with any system of teaching, when one uses inquiry-based learning methods (even those that are not modified Moore method based) it is important to remain practical

[30]This point is intended as a nod to my colleague, Professor Randy Schaeffer, who is very keen (correctly) and has argued for proficiency and not simply competency.

[31] Hopefully, this will change. The author has submitted two proposals to the department, one for a freshman course in logic (2 semester hours) and another to convert to the standard Calculus I – II - III (4 semester hours each) rather than the KUP mathematics department curriculum of 2009 which is Calculus I – II - III -IV (3 semester hours each).

[32] The majority of my colleagues employ traditional methods.

and patient.

Nonetheless, using such a method, it is opined, creates an atmosphere for scholarly inquiry and activity so that undergraduate research is initiated and incorporated into the undergraduate experience early in a student's mathematics programme and continues throughout his undergraduate years.

It has become rather accepted in the modern academy and throughout much of mathematics education to include in discourse a position statement or a statement from whence someone argues. Some educators argue that all knowledge is tinted by the background and perspective of the individual or more often from the group that person is a member of (be it religious, ethnic, etc.). This position is constructivism or radical constructivism (it has also been called phenomenology or hermeneutics). In such a schema, there is no global truth, all is relative, and the best one can hope for is a sharing of perspectives but no conclusion can necessarily be drawn. However, such a position belies the great work done by mathematicians throughout the centuries and negates the consequences of the discoveries, inventions, observations, and realisations that were created. There has to be some foundation of objectivism that underlies a proper philosophy of mathematical inquiry and thought.

Constructivist or reform methods include class discussions, use of technology, applications, modelling, and group assignments. The MMM includes class discussion and allows for the discussion to flow from the students but be directed by the instructor and as stated previously the MMM allows for applications and modelling.

Several authors submit a constructivist approach to the learning of, teaching about, or even doing mathematics [27, 28, 37, 40, 45, 68, 78, 80, 81, 84, 87][33] The constructivist accentuates the community and focuses on cooperation amongst learners. If one agrees with the philosophical position conditional to the constructivist method, then it may be an entirely acceptable learning or teaching methodology and might be a position grounded for a philosophy of inquiry in an area other than mathematics but it seems to be highly suspect as a philosophy of education of mathematical inquiry. This is because it seems of little practical use in the *doing* of mathematics and is not a *foundation* upon which conclusions can be drawn; hence, how would one be able to convey results, argue veracity, or generalise with any reliability? It seems that the constructivist method is best suited for elementary problems where inquirers have not completely matured and where the material is less sophisticated. The constructivist method is based on a philosophy that the individual learn with others and that reality is constructed. In its radical form it maintains "individuals construct their own reality through

[33]See especially Friere's, *The Politics of Education: Culture, Power, and Liberation,* 1985 or some of his other works for the foundation of constructivism along with Simon's, "Reconstructing Mathematics Pedagogy from a Constructivist Perspective," ERIC Document ED 364406.

actions and reflections of actions."[34] So, under such a philosophy a compleat relativism antecedes such that objectivism is relegated to oblivion. As a matter of the opinion of the author, constructivism seems to be a quite nihilistic, solipsistic, and a hopelessly subjective philosophical position. A constructivist negates the transcendent, universal, and objective nature of mathematics.

Recall 'positive scepticism' (or the principle of *epoikodomitikos skeptikistisis*) is meant to mean demanding objectivity; viewing a topic with a healthy dose of doubt; remaining open to being wrong; and, not arguing from an *a* priori perception. This need for some objectivism predicates the position the modified Moore method and inquiry-based learning is based (at least in part) on several traditions of philosophical thought. We depart from the classical philosophical position, call it $\mathcal{X}$, that that person M knows that thing p is true if and only if 1) M believes p; 2) p is true; and, 3) M is justified in believing that p is true. We shall call the position, call it $\mathcal{W}$, that person M knows that thing p is true if and only if 1) p is true and 2) M is justified in opining that p is true. That p is true implies that there is something that can be known apart from the individual M. That M is justified in opining that p is true requires a method of argument from the justification, requires that the justification be understandable, and that there was an accepted schemata employed for providing said justification. The author holds that belief is not a necessary condition for obtaining mathematical truth for it seems that belief is a consequent rather than an antecedent for knowing something and might not be needed even after obtaining knowledge.

A wonderful example is from Probability and Random Variable Theory. We know if X is a well-defined continuous random variable and the function given by $f(x)$ for each x in the domain of the function is the probability density function at x, then the expected value (or mean) of X is

$$E[X] = \int_{x} (x \cdot f(x))\, dx.$$

For $\alpha \in (0, \infty), \beta \in (0, \infty)$, then $X \sim Gamma(x, \alpha, \beta)$ where

$$Gamma(x, \alpha, \beta) = \frac{x^{(\alpha-1)}}{\Gamma(\alpha) \cdot \beta^{\alpha} \cdot e^{x/\beta}} \ni x \in (0, \infty)$$

whilst $0 \ni x \in (-\infty, 0]$ It is enjoyable to prove

$$E[X] = \alpha \cdot \beta.$$

However, for $\alpha \in (0, \infty), \beta \in (0, \infty)$, then $X \sim Cauchy(x, \alpha, \beta)$ where

$$Cauchy(x, \alpha, \beta) = \frac{\beta/\pi}{(x-\alpha)^2 + \beta^2} \ni x \in (-\infty, \infty).$$

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[34] Steffe and Kieren, "Radical Constructivism and Mathematics Education," Journal for Research in Mathematics Education 25, no. 6 (1994): 721.

It is most enjoyable to prove $E[X]$ does not exist! The student need *not* believe a result in order to *deduce* it or *know* it; hence, in mathematics we have $\mathcal{W}$.

Hence, IBL actualised by our MMM adopts a modicum of objectivism (there are ideals, there is a real world, it has meaning, and we can know some of the things that exist) along with a position that knowledge is gained through M.

We have mathematics which has universals; there exist principles to be discovered, created, or invented. It could be there are others in the universe or other universes; yet, they would have π (though perhaps of a different name but it would be π) and it exists as it exists here. This is quite a different situation than the social sciences, arts, humanities, etc. which hinge on a subjective slant and relative interpretation. We are not bound by the idea of *"interpreting"* the meaning of π, it simply *is*. This demonstrates that we can understand it but must also *get it right*.[35] Hence, the philosophy of mathematics is inexorably bound to the notion of being correct, of bounding error (when error exists), and of being able to note when we are wrong. The root of objectivism is fundamental to mathematical thought and inquiry-based learning.

A discussion of IBL versus traditionalism or constructivism would not be compleat without noting that a conflict that appears often in the literature and in academe is one between and betwixt 'mathematics education' and 'mathematics.' There seems to be a large philosophical void between 'mathematics educators' and 'mathematicians.' Oft 'mathematics educators' write of pedagogy and processes but not the content; whilst, 'mathematicians' spend time doing mathematics and have little (if any) interest in discussion of pedagogy and processes. We opine that inquiry-based learning (IBL) bridges that gap between the two for as a content-centred instructional methodology it forces one to focus on the content and how that content is communicated, described, and how learners are facilitated to learn.

[35] Conditionally: consequent to classical logic, Zermelo-Frankel-Cantor set Theory, and a set of axioms.

V. The Challenges with Inquiry-Based Learning (IBL) and Conclusion

If there was ***a*** way to teach mathematics, perhaps this paper would not exist. However, it seems commonly accepted that (a) different individuals learn in different ways and (b) there is a basic knowledge base that is necessary for the average student to obtain so that he has a higher likelihood to succeed in upper division courses. Hence, dogmatically approaching pedagogy be it either through the avenue of the Moore method, constructivism, traditionalism, or another method deprives both the instructor and the student of learning opportunities. It deprives the instructor by not allowing him to see that, 'only one way,' might not be correct. It deprives the student because his learning style might not be geared toward the approach taken by the instructor. Authentic IBL doesn't encourage or support pedagogical dogmatism. Much of the general educational research of the last thirty years centred on (a), thus we shall not bother wasting paper addressing in detail this point. Much work of professional associations (in particular the Mathematical Association of America (MAA)) research and policy statements of the last thirty years centred on defining (b) and revising, enhancing, and reviewing (b) [11 - 20, 64].

Let us grant that there an element of the traditionalist position is grounded; that is to say that skills can be mastered (though not definitively through exposition, modelling, practice, and hard work but through elements of each as well as perhaps more not listed). Proving claims true or false being a skill is grounded in the philosophy of William James and the practice of George Pólya. Just as art schools teach composition techniques, architecture schools teach drafting, etc. schools of mathematics teach theorem proving as a skill that is grounded in logic. There *are* a finite number of techniques and students are encouraged to learn each one so there is an element of the concept of basic competency that is reasonable when approaching mathematical claims.

A focal point of the discussion of the methods of inquiry-based learning (IBL) as enacted through a modified Moore method (MMM) is the uncompromising demand for justification. The MMM instructor must insist that his students (and he himself) justify every claim, every step of a proof when the proof is written formally, and explain to the students the rationale for such a policy. If one happens upon a fact but really does not know why the fact is indeed so, does he really know the thing he claims to know? The author is fond of quoting his late mother to the students, "mean what you say and say what you mean." The object of a lesson in the IBL classroom and the object of furthering an undergraduate's progress toward authentic understanding of mathematics research seem completely compatible; that is, to encourage thought, to encourage deliberation, to encourage contemplation, and to encourage a healthy dose of scepticism so that one does not wander too far into a position of subservience, 'give-me-the-answer'-ism, or

a position of arrogance, ‘know-it-all’-ism.

The MMM requires the instructor adopt an approach such that inquiry is ongoing. A demand for understanding what is and why it is, what is not known and an understanding of why it is not known, the difference between the two, and a confidence that if enough effort is exerted, then a solution can be reasoned. In this way, the MMM method is perhaps most similar to the Moore method. Consider:

> Suppose someone were in a forest and he noticed some interesting things in that forest. In looking around, he sees some animals over here, some birds over there,and so forth. Suppose someone takes his hand and says, ‘Let me show you theway,’ and leads him through the forest. Don’t you think he has the feeling that someone took his hand and led him through there? I would rather take my time and find my own way.[36]

However, it must be emphasised the confidence must be tempered with humility and realism. Hence, one must be selective; one must accept his limitations; and one must realise that not everything can be known.

The nature of conjectures arising from the students is in keeping with reform methods. However, as with the Moore method, the MMM instructor is free (and indeed should) pose pertinent questions to students which might not germinate from the students. Oft this is dependent on the nature of the composition of the student body taking the class in a particular semester. Thus, again it should be noted that the MMM requires as much flexibility as possible on the part of the instructor to gauge the mathematical maturity of the class members and adjust accordingly.[37] Each semester brings with it new students and so new challenges. It is incumbent upon the instructor to keep vigil and assess the progress of the students.

I am also more convinced now than probably at any point previous that Moore was right - in the competition between Sophistry and Socraticism, Socraticism is authentic, preferred, and correct. However, Sophistry is ascendant in the 21^{st} century school, academy, institute, college, and university. It prevails in many a classroom because:
1) it is easier for the instructor–no arguments with students, parents, or administrators; complaints of things being ’hard’ are almost non-existent if one employs sophistry and the instructor does not have to ”think as hard;”
2) it is easier for the student–he does not have to ”think as hard” (or think at all), she can ”feel good,” it can have its self-esteem ’boosted;’
3) it is easier for the institution– standardisation can be employed (which seems to be a goal at many institutions); students retained (’retention’ seems

[36] Moore, *C*hallenge in the Classroom.
[37] Adjust ***does not imply*** ‘dumb down.’

to be a 21^{st} 'buzz word'), graduation rates increase, and accrediting agencies are mollified (such as the Middle States Association of Colleges and Schools ('Middle States'), the Southern Association of Colleges and Schools ('SACS'), or National Council for the Accreditation of Teacher Education ('NCATE')).[38]

If a Socratic were questioning the worth of a programme or its effectiveness then none of the aforementioned considerations would be primary. For a Socratic, we would be concerned with measuring a course, the instructor, the content, etc. of a course let us call it $\mathcal{A}$, by how well a student does in a course or courses subsequent to $\mathcal{A}$ for which $\mathcal{A}$ is an authentic pre-requisite or to which it applies (say in a different field). For a Socratic, we would be concerned with measuring a programme by how well its graduates succeed in graduate school, the workforce, etc. Such is not how *it is, but how it, perhaps, should be.*

This paper simply offers a view as to an aspect of mathematical inquiry which the author has come to opine might be part of that which works to assist student learning. Traditionalism, we opine, is too rigid, inflexible, and static such that knowledge is bestowed rather than learnt. Phenomenology, hermeneutics, radical constructivism, and their kin see mathematics primarily as a social construct, as a product of culture, subject to interpretation, and it changes dependent on audience. Like the other sciences, mathematics is viewed as empirical endeavours whose results are constantly compared to reality and may be discarded when not in agreement with observation, seem pointless, or are 'too abstract.' The belief that mathematics is hounded by the fashions of the social group performing it or by the needs of the society financing it seems meritless given the permanence of mathematics.

What binds and supports mathematics is a search for truth, a search for what works, and a search for what is applicable within the constraints of the demand for justification. It is not the ends, but the means which matter the most - - the process at deriving an answer, the progression to the application, and the method of generalisation. These procedures demand more than mere speculative ideas; they demand reasoned and sanguine justification. For a student to understand that student must opine, hypothesise, conjecture, describe the process of deriving an answer, explain the progression to the application, justify the method of generalisation - - these actions demand more than mere 'mathematics appreciation,' 'busy work' activities,

[38] In my career I have now been privy to and a participant in 3 decennial university re-accreditation processes and have found the processes lacking in any authentic search for improvement in academics. It seems to be a public relations operation on the part of the school and a burdensome set of bureaucratic hoops that were created and expanded on the part of the accrediting agency for the members of the university to have to 'jump through' in order to be re-accredited. I understand that there are instances where accrediting agencies are needed to insure that academia is not compromised (academia can be compromised such as was the case with Morris Brown College) but there also seems to be decennial 'games' being played and many criteria established by accrediting agencies seem to obfuscate, complicate, and not clarify the mission of a department, college or university or create a reason for the existence of the accerediton body (the raison d'étre for the exercise)).

or lectures - - they require objectivity; positive scepticism; and, a willingness to be wrong.

The author tries to include the Halmos quote of Moore in each paper he writes (sometimes it is not included in the most elegant of ways). P. J. Halmos recalled a conversation with R. L. Moore where Moore quoted a Chinese proverb. That proverb provides a summation of the justification of the methods employed in teaching students to do mathematics by using IBL techniques as executed by our MMM and provides incite into the foundation of the MMM. It states, "I see, I forget; I hear, I remember; I *do* , I *understand*."

It is in that spirit that a core point of the argument presented in the paper is that inquiry-based learning (IBL) is a pedagogical position that deviates from the 'norm' insofar as it argues <u>the content studied</u> is the keystone which holds an academic endeavour together; it is the reason for academe to exist; it is the reason for inquiry – the student and the instructor should be and are secondary to the material in a university setting. The experience of **doing** mathematics rather than **witnessing** mathematics is that which we, a mathematics faculty, need encourage; for it is that which we love to do as well - math!

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